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The shape of a magnetic liquid drop

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The electromagnetic forces in a ferrofluid depend on the domain occupied by the fluid. We study here the equilibrium positions of a ferrofluid drop with a boundary which is partially or totally free. The method used is based on the minimization of the energy with respect to the shape of the drop. We show bifurcations of the solutions and hysteresis phenomena when the parameters vary.

1. Introduction

The problem of the equilibrium shape of a liquid with non-local forces is difficult. Non-local forces means here that the forces depend on the shape of the domain occupied by the fluid as in the case of magnetic fluids, molten metals or bodies with a self-gravity field. This coupling can produce instabilities or a non-trivial hydrostatic problem. For magnetic liquids a large literature exists on the subject, see for instance, Blums, Maiorov & Tsebers (1989) and Rosensweig (1985). Sneyd & Moffatt (1982), for molten metals, and Brancher & Séro-Guillaume (1983), for magnetic liquids, derived a variational principle for investigating equilibria of such fluids. In some sense the variable is the position of the domain, and the equilibrium is reached when the energy is extremum with respect to this variable. The variations of energy are induced by variations of the boundary of the domain. Whether this extremum is a maximum or a minimum was not completely clear. But considering the dynamical problem, and assuming that the velocity field is potential, Séro-Guillaume & Bernardin (1987) have shown that the whole system of equations could be recast in Hamiltonian form. The variations are taken with respect to the value of the velocity potential on the free surface and to the position of the free surface. In a certain sense the free surface is a variable conjugate to this potential. The Hamiltonian is the sum of the kinetic energy and of the previously considered energy. Therefore this energy is a potential energy and has to be minimized to obtain equilibrium positions. This strategy had been proposed in Sneyd & Moffatt (1982), Brancher & Séro-Guillaume (1985) but it has been only used for two-dimensional configurations, for which the machinery of complex variables can be used.

Here we intend to determine equilibrium positions of axisymmetrical drops. Although the configurations are still two-dimensional, the method would be extendable to the full three-dimensional case.

In §2 we shall recall the equations of the problem in Newtonian and Hamiltonian form. From the Hamiltonian formulation we shall recover the derivative, or the variations of the potential energy with respect to variation of the surface. This derivative is expressed as a surface integral which involves the interface normal displacement.

In §3, we deduce, from the energy derivative, a steepest descent algorithm. This algorithm is extended to the case of a sessile drop. The interaction of the drop with its support is simply modelled by an interfacial coefficient, which leads to the Young's condition for the contact angle.

The magnetic self-field potential and its normal derivative are calculated, in §4, by a boundary-integral method on the basis of a representation formula for a transmission problem.

The last section is devoted to numerical results. We consider a ferrofluid with a linear magnetization law and we present the results for the free drop and for the sessile drop. The shape of the drop without gravitation is closed to an ellipsoid although the curvature can be very different. In the case of the sessile drop we can detect the bifurcation and see the appearance of hysteresis. This bifurcation is characterized by peaks or waves which appear at the interface. We shall calculate the wavelength, by a linear analysis, of the peaks for the case of a strip of ferrofluid and we shall compare it with the one obtained by computation.

2. Dynamic and magnetic equations

Let us recall that a ferrofluid is a ferrite homogeneous suspension. When it is subjected to a magnetic field, say $\boldsymbol{H}_0$, a self-field $\boldsymbol{h}$ is created, which depends upon the domain Ω_1 occupied by the fluid; Ω_2 will be the exterior of Ω_1. We shall consider a simple isothermal model of magnetic liquids, see Rosensweig (1985), Brancher (1988), where the magnetization $\boldsymbol{M}$ is parallel to the total magnetic field $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}$, i.e. $\boldsymbol{M} = \chi(H)\,\boldsymbol{H}$.

2.1. *Mechanical equations*

The momentum equation and the conservation of volume can be written

$$\rho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = -\nabla(p+\rho g z)+\mu_0\boldsymbol{M}\cdot\nabla\boldsymbol{H}, \tag{2.1a}$$

$$\nabla\cdot\boldsymbol{V} = 0. \tag{2.1b}$$

Note that, as $\boldsymbol{M}$ and $\boldsymbol{H}$ are parallel,

$$\boldsymbol{M}\cdot\nabla\boldsymbol{H} = \nabla\left(\int_0^H M(y)\,\mathrm{d}y\right). \tag{2.1c}$$

The surface stress density is

$$\boldsymbol{T} = -p\boldsymbol{n}-\tfrac{1}{2}\mu_0(\boldsymbol{M}\cdot\boldsymbol{n})^2\boldsymbol{n} = -p\boldsymbol{n}-\tfrac{1}{2}\mu_0 M_n^2\boldsymbol{n} \tag{2.2}$$

where $\boldsymbol{n}$ is the outward unit normal to the boundary S of Ω_1, and M_n is the scalar product in Euclidean space, of $\boldsymbol{M}$ and $\boldsymbol{n}$. If we suppose that the velocity is potential, i.e.

$$\boldsymbol{V} = \nabla\varphi,$$

then (2.1b) implies

$$\nabla^2\varphi = 0 \quad \text{in} \quad \Omega_1,$$

and (2.1a), (2.1c), and (2.2) give the Bernoulli relation

$$\rho\frac{\partial\varphi}{\partial t}+\tfrac{1}{2}\rho|\nabla\varphi|^2+p+\rho g z-\mu_0\int_0^H M(y)\,\mathrm{d}y = k(t). \tag{2.3}$$

We suppose that the exterior fluid is at a constant pressure, then with (2.2), (2.3) and Laplace law we can write:

$$\rho\frac{\partial\varphi}{\partial t}+\tfrac{1}{2}\rho|\nabla\varphi|^2+\rho g z-\tfrac{1}{2}\mu_0 M_n^2-\mu_0\int_0^H M(y)\,\mathrm{d}y+\sigma C=k(t). \tag{2.4}$$

C is the mean curvature of S and σ the interfacial coefficient. Let us write now that the normal velocity of S is equal to the normal velocity of the fluid, i.e.

$$\frac{\partial\theta}{\partial t}=\frac{\partial\varphi}{\partial n}\quad\text{on}\quad S, \tag{2.5}$$

where $x=\theta(a,t)$ is the Lagrangian position of the fluid particles.

2.2. *Magnetic equations*

Now if $\boldsymbol{H}_i=\boldsymbol{H}_0+\boldsymbol{h}_i$, in Ω_i, let us note that

$$\nabla\wedge\boldsymbol{h}_i=0,$$

and then $\boldsymbol{h}_i=\nabla\boldsymbol{u}_i$ in Ω_i. The magnetic inductions are $\boldsymbol{B}_1=\mu_0(\boldsymbol{H}_1+\boldsymbol{M}(H_1))$ in Ω_1, and $\boldsymbol{B}_2=\mu_0\boldsymbol{H}_2$ in Ω_2. They satisfy $\nabla\cdot\boldsymbol{B}_i=0$ in Ω_i. The normal component of $\boldsymbol{B}$ and the tangential component of $\boldsymbol{H}$ are conserved through S, i.e. $H_{2n}=H_{1n}+M_n$ and $H_{2t}=H_{1t}$. Using the scalar potential u_i, all these relations can be written

$$\nabla^2 u_1=-\nabla\cdot\boldsymbol{M}\quad\text{in}\quad\Omega_1, \tag{2.6a}$$

$$\nabla^2 u_2=0\quad\text{in}\quad\Omega_2, \tag{2.6b}$$

$$u_1=u_2\quad\text{on}\quad S, \tag{2.6c}$$

$$\frac{\partial u_1}{\partial n}-\frac{\partial u_2}{\partial n}=-M_n\quad\text{on}\quad S. \tag{2.6d}$$

The problem is to find S, φ, u_1, u_2 which satisfy the relations (2.3), (2.4), (2.5), (2.6).

Let us consider the magnetic coenergy:

$$E_{\mathrm{m}}=\tfrac{1}{2}\mu_0\int_{\Omega_1}\left(H_1^2+2\int_0^{H_1}M(y)\,\mathrm{d}y\right)\mathrm{d}\Omega+\tfrac{1}{2}\mu_0\int_{\Omega_2}H_2^2\,\mathrm{d}\Omega. \tag{2.7}$$

This coenergy is such that a variation $\delta\boldsymbol{H}$ produces a variation in E_{m} given by

$$\delta E_{\mathrm{m}}=\int\boldsymbol{B}\cdot\delta\boldsymbol{H}\,\mathrm{d}\Omega.$$

We also consider the gravitational, interfacial, and kinetic energy:

$$U_{\mathrm{g}}=\int_{\Omega_1}\rho g z\,\mathrm{d}\Omega,\quad U_\sigma=\sigma\int_S\mathrm{d}S,\quad K=\int_{\Omega_1}\tfrac{1}{2}\rho|\nabla\varphi|^2\,\mathrm{d}\Omega \tag{2.8}$$

and the Hamiltonian:

$$\mathscr{H}=\frac{1}{\rho}(K+U_{\mathrm{g}}+U_\sigma-E_{\mathrm{m}}). \tag{2.9}$$

The value φ_S on the surface of the velocity potential, and the surface S can be considered as conjugate variables, see Séro-Guillaume & Bernardin (1987), and Séro-

Guillaume & Brancher (1991) for a general treatment. Considering normal variations $\delta\theta$ of the interface, the mechanical equations of the problem can be recast in the following Hamiltonian form

$$\frac{\partial\theta}{\partial t}=\frac{\delta\mathscr{H}}{\delta\varphi_S};\quad \frac{\partial\varphi_S}{\partial t}=-\frac{\delta\mathscr{H}}{\delta\theta}. \tag{2.10a, b}$$

We shall examine in detail at the next section the meaning of relations (2.10). Nevertheless, we can conclude now that the functional

$$\mathscr{V}=U_g+U_\sigma-E_m, \tag{2.11}$$

is a real potential with respect to the positions and shapes of the surface S. Therefore the stable equilibrium positions of the system are obtained with the domain Ω_1 (and then Ω_2) which minimizes $\mathscr{V}$.

One can consider that the minus sign before E_m is a paradox. Let us recall that if E_m^* is the energy of the system, i.e. is such that

$$\delta E_m^*=\int \boldsymbol{H}\cdot\delta\boldsymbol{B}\,\mathrm{d}\Omega,$$

one can show, see Brancher & Séro-Guillaume (1985), that the force is given by $\boldsymbol{F}=\boldsymbol{\nabla}(E_m)=-\boldsymbol{\nabla}(E_m^*)$, the gradient of E_m is obtained by taking its variation with respect to the position of the particles, but keeping the current in the inductor fixed. For more details see the Appendix.

3. Steepest descent algorithm

3.1. *The free drop*

In fact the relations (2.10) are equalities between linear functionals: (2.10a) means

$$\int_S \frac{\partial\theta}{\partial t}\delta\varphi_S\,\mathrm{d}S=\frac{\delta\mathscr{H}}{\delta\varphi_S}(\delta\varphi_S)=\int_S\frac{\partial\varphi}{\partial n}\delta\varphi_S\,\mathrm{d}S$$

for any variations $\delta\varphi_S$; and (2.10b) means

$$\int_S\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial n}\frac{\partial\theta}{\partial t}\right)\delta\theta\,\mathrm{d}S=-\frac{\delta\mathscr{H}}{\delta\theta}(\delta\theta),$$

with

$$\frac{\delta\mathscr{H}}{\delta\theta}(\delta\theta)=\int_S\left(\frac{1}{2}|\boldsymbol{\nabla}\varphi|^2+gz-\frac{1}{2\rho}\mu_0M_n^2-\frac{\mu_0}{\rho}\int_0^{H_1}M(y)\,\mathrm{d}y+\frac{\sigma}{\rho}C-\left(\frac{\partial\varphi}{\partial n}\right)^2\right)\delta\theta\,\mathrm{d}S. \tag{3.1}$$

This relations holds for any $\delta\theta$ such that

$$\int_S\delta\theta\,\mathrm{d}S=0, \tag{3.2}$$

because of volume conservation. We have to minimize the potential defined by (2.11) under the volume conservation constraint; therefore let us consider the Lagrange parameter λ and the modified potential:

$$\mathscr{V}_1=\mathscr{V}+\lambda|\Omega_1|,$$

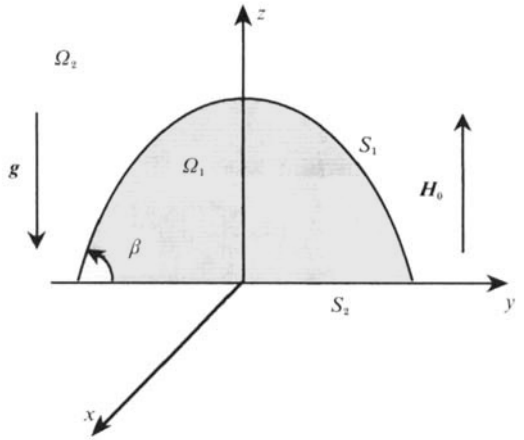

FIGURE 1. The parameters and coordinate system for a sessile drop.

$|\Omega_1|$ is the volume of Ω_1. Setting φ equal to zero in (3.1) we can easily deduce the variations of $\mathscr{V}_1$ with respect to any variations $\delta\theta$ of S, i.e. the $\delta\theta$ may have tangential components on S (for a direct calculation see Brancher & Séro-Guillaume 1985):

$$\delta\mathscr{V}_1 = \int_S \left(\rho g z + \sigma C - \tfrac{1}{2}\mu_0\left(M_n^2 + 2\int_0^{H_1} M(y)\,\mathrm{d}y\right) + \lambda\right)\delta\boldsymbol{\theta}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{3.3}$$

Relation (3.3) gives the steepest descent direction for minimizing $\mathscr{V}_1$, which is

$$\delta\boldsymbol{\theta}_{\mathrm{op}} = -\left(\rho g z + \sigma C - \tfrac{1}{2}\mu_0\left(M_n^2 + 2\int_0^{H_1} M(y)\,\mathrm{d}y\right) + \lambda\right)\boldsymbol{n}, \tag{3.4}$$

the parameter λ, as we shall see later, has to be calculated with the help of condition (3.2).

To decrease the potential we have to move each surface point in the direction given by (3.4), and thus we deduce the following algorithm:

$$\left.\begin{aligned} &\Omega_1^0, \Omega_2^0, S^0 (S^0 = \partial\Omega_1^0) \text{ given,} \\ &S^{k+1} = S^k + \epsilon\delta\theta^k \quad \text{with} \quad \partial\Omega_1^{k+1} = S^{k+1}, \end{aligned}\right\} \tag{3.5}$$

$\delta\theta^k$ is calculated using (3.4) where all the quantities are evaluated by solving (2.6*a–d*) on Ω_i^k, then λ^k is given by

$$\lambda^k = -\frac{1}{|S^k|}\int_{S^k}\left(\rho g z + \sigma C - \tfrac{1}{2}\mu_0\left(M_n^{k2} + 2\int_0^{H_1^k} M^k(y)\,\mathrm{d}y\right)\right). \tag{3.6}$$

ϵ is a real positive parameter determined in such a way that the value of the potential decreases.

3.2. *The sessile drop*

Let us suppose now that the drop is lying on non-ferromagnetic horizontal plane Σ. The boundary is now composed of two surfaces S_1, S_2, see figure 1; S_2 is the horizontal part of the interface, and $L = S_1 \cap S_2$ is the triple line.

We must now consider a potential which takes into account the interaction between the drop and the plane. Let $\sigma_1, \sigma_2', \sigma_2''$ be the coefficients respectively of the ferrofluid interactions with the non-ferromagnetic fluid, with the solid non-ferromagnetic plane and between the plane and the non-ferromagnetic fluid. The total interfacial energy is now

$$U_\sigma = \int_{S_1} \sigma_1 \,\mathrm{d}S + \int_{S_2} \sigma_2' \,\mathrm{d}S + \int_{\Sigma - S_2} \sigma_2'' \,\mathrm{d}S.$$

This potential up to an additive constant is given by

$$U_\sigma = \int_{S_1} \sigma_1 \,\mathrm{d}S - \int_{S_2} \sigma_2 \,\mathrm{d}S, \tag{3.7}$$

with $\sigma_2 = \sigma_2'' - \sigma_2'$.

We consider again the potential energy, with the interfacial energy given by (3.7). The admissible displacements or the interface variations on S_2 and L must be done along the plane Σ. The variations of the potential are now

$$\delta \mathscr{V} = \int_{S_1} \left(\rho g z + \sigma C - \tfrac{1}{2}\mu_0 \left(M_n^2 + 2 \int_0^{H_1} M(y)\,\mathrm{d}y \right) + \lambda \right) \delta\boldsymbol{\theta} \cdot \boldsymbol{n} \,\mathrm{d}S$$

$$- \int_L \boldsymbol{t} \times (\sigma_1 \boldsymbol{n}_1 - \sigma_2 \boldsymbol{k}) \cdot \delta\boldsymbol{\theta} \,\mathrm{d}l, \tag{3.8}$$

$\boldsymbol{t}$ is a unit tangent vector to L, $\boldsymbol{n}_1$ is a unit normal vector to S_1 and $\boldsymbol{k}$ is a unit upward vector. As it can be seen, the variations (3.8) are null for all admissible displacements $\delta\theta$ if and only if

$$G(\Omega_1) = \rho g z + \sigma C - \tfrac{1}{2}\mu_0 \left(M_n^2 + 2 \int_0^{H_1} M(y)\,\mathrm{d}y \right) = c \quad \text{on} \quad S_1 - L. \tag{3.9}$$

Here c is a constant, and

$$\sigma_2 = \sigma_1 \cos\beta, \tag{3.10}$$

where β is the contact angle, see figure 1. The new condition (3.10) is nothing but the Young's boundary condition.

3.3. *Minimization algorithm*

Here again we consider the same modified potential, and we construct the new surface S^{k+1} in such a way that:

$$(S^{k+1} S^k + \epsilon(G^k(\Omega_1^k) + \lambda^k)$$

and each point of L^k is translated by a vector such that its normal component is

$$-\epsilon(\sigma_2 - \sigma_1 \cos\beta).$$

4. Computation of the self-field

We shall consider a ferrofluid with a linear magnetization law $\boldsymbol{M} = \chi_0 \boldsymbol{H}$; then the Maxwell's system (2.6*a*–*d*) with an external uniform field $\boldsymbol{H}_0$ reduces to:

$$\nabla^2 u_1 = 0 \quad \text{in} \quad \Omega_1, \tag{4.1a}$$

$$\nabla^2 u_2 = 0 \quad \text{in} \quad \Omega_2, \tag{4.1b}$$

$$u_1 = u_2 \quad \text{on} \quad S, \tag{4.1c}$$

$$(1+\chi_0)\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = -\chi_0 \boldsymbol{H}_0 \cdot \boldsymbol{n} \quad \text{on} \quad S, \tag{4.1d}$$

$$u_2 = O(1/|x|) \quad \text{at infinity.} \tag{4.1e}$$

In this case the magnetic energy can be written, cf. Zouaoui (1991),

$$E_{\mathrm{m}} = \tfrac{1}{2}\mu_0 \chi_0 H_0 \int_{\partial\Omega_1} u_1 \boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{4.2}$$

with $\boldsymbol{H}_0 = H_0 \boldsymbol{k}$, where $\boldsymbol{k}$ is a vertical upward vector.

We shall also use a reduced potential, in terms of V_0, the volume of the drop:

$$\mathscr{V}^* = B_{\mathrm{g}} \int_{\Omega_1^*} z^*\,\mathrm{d}\Omega + \int_{S_1^*} \mathrm{d}S - \sigma_2^* \int_{S_2^*} \mathrm{d}S - \tfrac{1}{2}\chi_0 B_{\mathrm{m}} \int_{S_1^* \cup S_2^*} u^* \boldsymbol{k}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{4.3a}$$

with

$$u^* = \frac{u}{H_0 V_0^{\frac{1}{3}}}, \quad z^* = \frac{z}{V_0^{\frac{1}{3}}}, \quad \sigma_2^* = \frac{\sigma_2}{\sigma_1}, \tag{4.3b}$$

$$S_i^* = \frac{S_i}{V_0^{\frac{2}{3}}} \quad (i = 1, 2), \quad \mathscr{V}^* = \frac{\mathscr{V}}{\sigma_1 V_0^{\frac{2}{3}}}, \tag{4.3c}$$

$$B_{\mathrm{m}} = \frac{\mu_0 H_0 V_0^{\frac{1}{3}}}{\sigma_1}, \quad B_{\mathrm{g}} = \frac{\rho g V_0^{\frac{2}{3}}}{\sigma_1}. \tag{4.3d}$$

B_{m} and B_{g} are respectively the magnetic and the gravitational Bond numbers.

The relations (4.1), (4.3) correspond to the case of a sessile drop; to recover the case of a free drop, it is sufficient to set B_{g} equal to zero, and to make the interface S_2 vanish.

The magnetic potential will be calculated by the boundary-integral technique, cf. Brebbia, Telles & Wrobel (1984). Let us first derive an integral equation for the potential and its normal derivative.

Let u^* be a radial fundamental solution of the Laplace equation, i.e. u^* satisfies

$$\nabla_y^2 u^*(x, y) = -\delta_x(y),$$

where $\delta_x(y)$ is the Dirac distribution concentrated at x. It is known that if u is harmonic in the domain Ω, then u satisfy the following integral equation:

$$c(x)\,u(x) = \int_{\partial\Omega} \left(u^*(x,y)\frac{\partial u}{\partial n_y} - u(y)\frac{\partial u^*(x,y)}{\partial n_y} \right) \mathrm{d}S(y), \tag{4.4}$$

with
$$c(x) = -\frac{\omega(x)}{4\pi} = \int_{\partial Q} \frac{\partial u^*(x,y)}{\partial n_y}\,\mathrm{d}S(y).$$

$\partial u^*/\partial n_y$ is the normal derivative with respect to y. In fact $\omega(x)$ is the solid angle under which the surface $\partial\Omega$ is seen from x. Therefore $c(x)$ is equal to 0 if x is outside Ω, $\frac{1}{2}$ if x is on a regular point of $\partial\Omega$ (i.e. a point where $\partial\Omega$ has a tangent plane), and 1 if x is in Ω.

If $u = (u_1, u_2)$ satisfies the system (4.1), for x on the boundary $\partial\Omega$, the potential and its normal derivative $q(x)$ satisfy the following relations:

$$k(x)\,u(x) + \int_{\partial\Omega_1} u(y)\,q^*(x,y)\,\mathrm{d}S(y) = -\int_{\partial\Omega_1} \boldsymbol{H}_0\cdot\boldsymbol{n}u^*(x,y)\,\mathrm{d}S(y), \tag{4.5a}$$

$$\int_{\partial\Omega_1} u^*(y)\,q(x,y)\,\mathrm{d}S(y) = -\frac{1}{\chi_0}u(x) - \int_{\partial\Omega_1} \boldsymbol{H}_0\cdot\boldsymbol{n}u^*(x,y)\,\mathrm{d}S(y), \tag{4.5b}$$

with $k(x) = (1/\chi_0) - (\omega(x)/4\pi)$; and q^* is the normal derivative of u^*.

To obtain the above relation let us suppose that u_2 is known, then u_1 will satisfy the following system:

$$\nabla^2 u_1 = 0 \quad \text{in} \quad \Omega_1, \tag{4.6a}$$

$$u_1 = u_2 \quad \text{on} \quad \partial\Omega_1. \tag{4.6b}$$

And if u_1 is known we have the following exterior problem for u_2:

$$\nabla^2 u_2 = 0 \quad \text{in} \quad \Omega_2, \tag{4.7a}$$

$$\frac{\partial u_2}{\partial n} = q_2 \quad \text{on} \quad \partial\Omega_2, \tag{4.7b}$$

$$u_2 = O\left(\frac{1}{x}\right) \quad \text{and} \quad \nabla^2 u_2 = O\left(\frac{1}{|x|^2}\right) \quad \text{at infinity}. \tag{4.7c}$$

We can apply the integral formula (4.4) to the solutions of (4.6) and (4.7):

$$c(x_1)\,u_1(x_1) = \int_{\partial\Omega_1} (u_1^*(x_1, y)\,q_1(y) - u_1(y)\,q_1^*(x_1, y))\,\mathrm{d}S(y), \tag{4.8a}$$

$$c'(x_2)\,u_2(x_2) = \int_{\partial\Omega_2} (u_2^*(x_2, y)\,q_2(y) - u_2(y)\,q_2^*(x_2, y))\,\mathrm{d}S(y). \tag{4.8b}$$

Because of the condition (4.7c), the integral in (4.8b) reduces to an integral on $\partial\Omega$ only. If we take into account (4.1c) and (4.1d), (4.8b) becomes

$$c'(x_2)\,u_2(x_2) = (1+\chi_0)\int_{\partial\Omega_1} u_1^*(x_2, y)\,q_1(y)\,\mathrm{d}S(y) - \int_{\partial\Omega_1} u_1(y)\,q_1^*(x_1, y)\,\mathrm{d}S(y) - \chi_0 \int_{\partial\Omega_1} \boldsymbol{H}_0\cdot\boldsymbol{n}u_2^*(x_2, y)\,\mathrm{d}S(y). \tag{4.9}$$

If we set $x_1 = x_2 = x$, then $n_1 = -n_2$, $q_1^* = -q_2^* = q^*$, and $c(x)+c'(x) = 1$. Therefore (4.8a) can be written

$$c(x)\,u(x) = \int_{\partial\Omega_1} u^*(x, y)\,q(y)\,\mathrm{d}S(y) - \int_{\partial\Omega_1} u(y)\,q^*(x, y)\,\mathrm{d}S(y) \tag{4.10}$$

and (4.9) becomes

$$(1-c(x))\,u(x) = -(1+\chi_0)\int_{\partial\Omega_1} u^*(x, y)\,q(y)\,\mathrm{d}S(y) + \int_{\partial\Omega_1} u(y)\,q^*(x, y)\,\mathrm{d}S(y) - \int_{\partial\Omega_1} \boldsymbol{H}_0\cdot\boldsymbol{n}u^*(x, y)\,\mathrm{d}S(y). \tag{4.11}$$

If we multiply (4.10) by $(1+\chi_0)$ and add the result to (4.11) we obtain (4.5a). Finally the sum of (4.10) and (4.11) gives, up to χ_0, (4.5b).

The surface $\partial\Omega_1$ has a meridian line composed of two curves C_1 and C_2. Of course

C_2 does not exist for the case of the free drop, and it is a line segment for the case of a sessile drop. These two curves can be divided into intervals (M_i, M_{i+1}) and the functions u and q are interpolated by polynomial expressions on each interval. The unknowns are now the values u_i, q_i of u and q on each node M_i. Then (4.5*a*) (4.5*b*) are transformed into the following algebraic system, see Zouaoui (1991):

$$k_i u_i + \sum_{j=1}^{N} h_{ij} u_j = b_i, \tag{4.12a}$$

$$\sum_{j=1}^{N} g_{ij} q_j = -\frac{1}{\chi_0} u_i + c_i. \tag{4.12b}$$

This system of $2N$ equations with $2N$ unknown is solved by the Gaussian elimination method.

5. Results and discussion

The numerical calculus of the field has been tested with ellipsoidal shapes, because the exact solution is known in this case. For 10 points of discretization the difference is less than 10^{-7}. The meridian curve, and therefore the normal vector, are interpolated by B-splines, see De Boor (1987). Let us describe some features of the numerical treatment:

For optimizing the computation time, the parameters B_m and B_g are slowly increased step by step. The initial shape is the preceding calculated equilibrium shape. The parameter ϵ in (3.5) is initialized at unity, and is divided by two if the energy has not decreased. If ϵ becomes small (typically $< 10^{-5}$) but the maximum value of $\delta\theta_{op}$ given by (3.4) remains greater than 5×10^{-2} we increase the number of discretization points, and this number varies from 40 to 400 for shape with rapidly varying curvature. Equilibrium is reached if $\delta\theta_{op}$ is smaller than 5×10^{-2}.

It is known, see Brancher & Zouaoui (1987) that when the susceptibility χ_0 is of order one, ellipsoids are almost exact solutions. We have compared our solutions (for the free drop) with the solutions obtained by ellipsoids, and the agreement is good. However, when the curvature is rapidly varying, a kind of interfacial wave (small amplitude and small wavelength) can appear. These waves produce small variations of the shape but high oscillations of the curvature. This phenomenon takes place near equilibrium. Magnetic terms have small variations with respect to these waves; therefore, considering that all terms are constant except the curvature in the equilibrium equation, we numerically inverse, by Newton's method, the curvature operator. After only one iteration the surface is smooth and the equilibrium equation is satisfied.

5.1. *The free drop*

In figure 2 we can see different shapes of the drop for increasing values of B_m. Berkovsky & Kalikmanov (1985) have studied the minimum of the energy in the set of ellipsoids. We can compare the calculated shapes with their results. For small values of B_m the meridian line is very close to an elliptic curve; but when B_m increases, the top of the drop becomes sharper and the mean curvature at this point differs appreciably from what it would be for an ellipse.

In figure 3 the ratio of the length to the breadth is plotted for different values of the susceptibility χ_0.

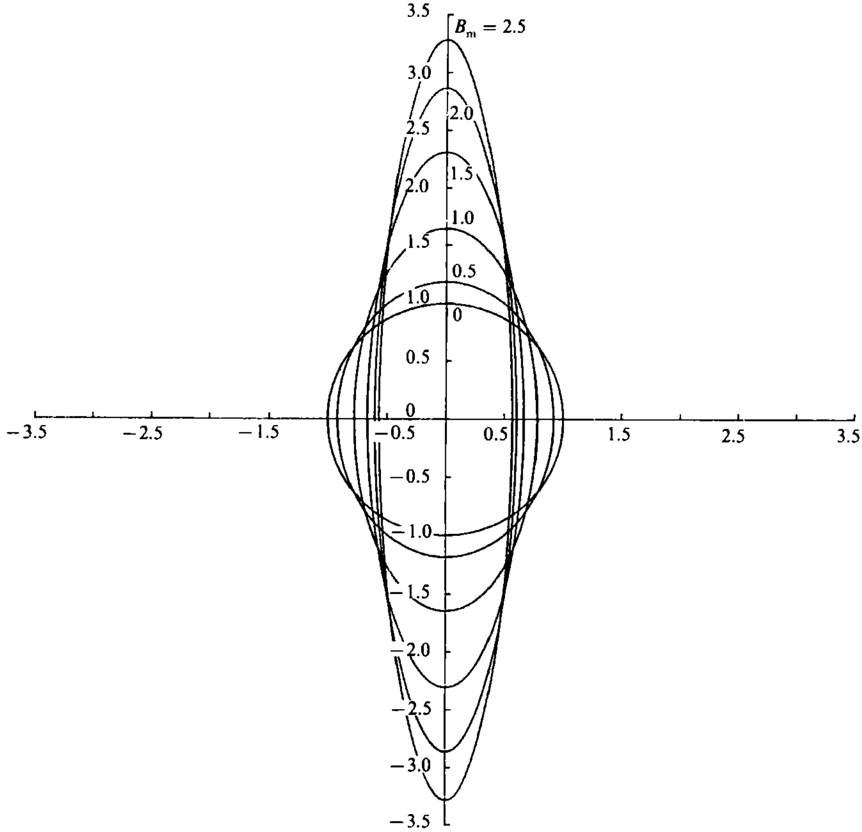

FIGURE 2. Equilibrium shapes of a free drop for different values of B_m.

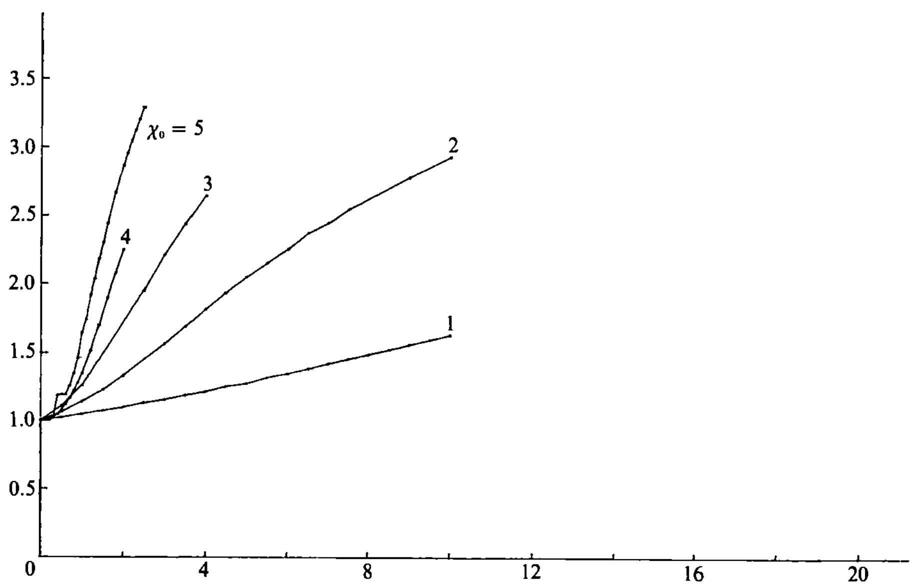

FIGURE 3. Ratio of the length to the breath of a free drop, for different values of χ_0.

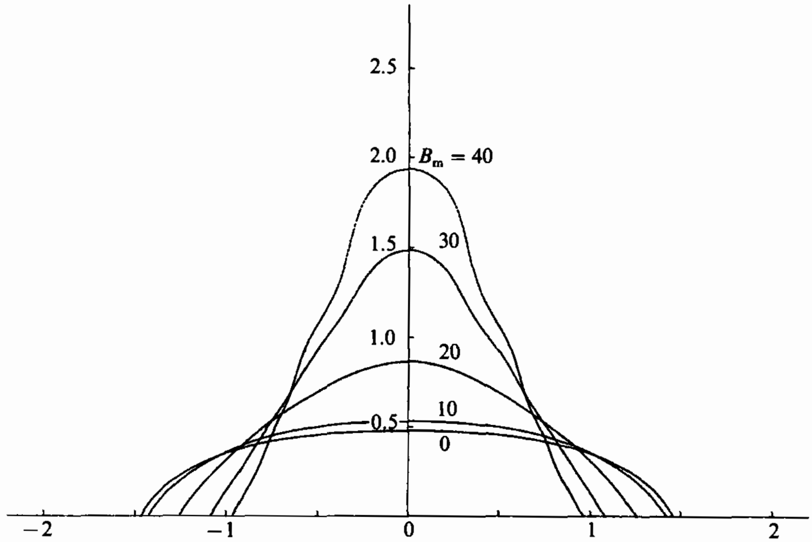

FIGURE 4. Several shapes of a sessile drop for B_g below the critical value.

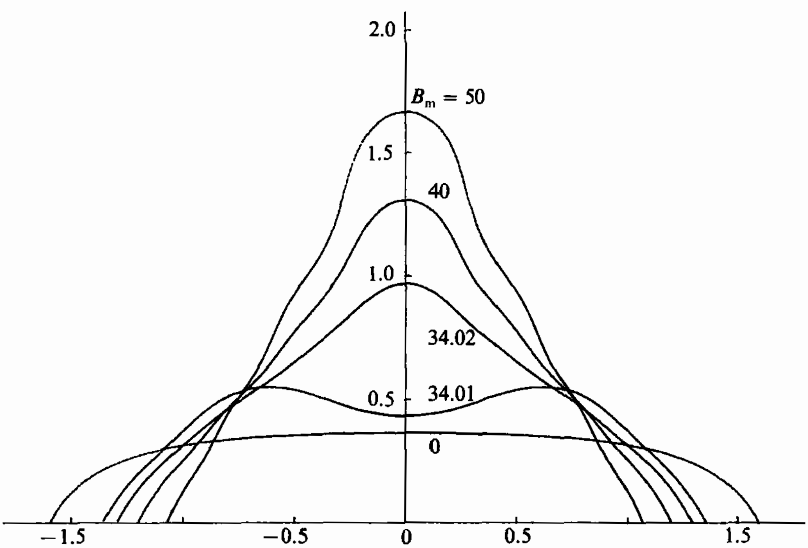

FIGURE 5. Several shapes of a sessile drop for B_g above the critical value.

5.2. *The sessile drop*

We shall focus on the wetting drop case, that is when the angle β is less than $\frac{1}{2}\pi$. For B_g lower than a critical value the height of the drop is an increasing function of B_m. There is an inversion of the curvature for B_m great enough, see figure 4 where various shapes are drawn for $B_g = 5$.

If B_g exceeds the critical value, some peaks appear on the drop, and when B_m increases, the centre of the drop jumps suddenly, see figure 5.

This can be seen more clearly on figures 6 and 7. In figure 6 the height of the drop is plotted versus B_m for different values of B_g. The jump between two equilibrium positions corresponds to a hysteresis phenomenon, which appears for B_g around 10.

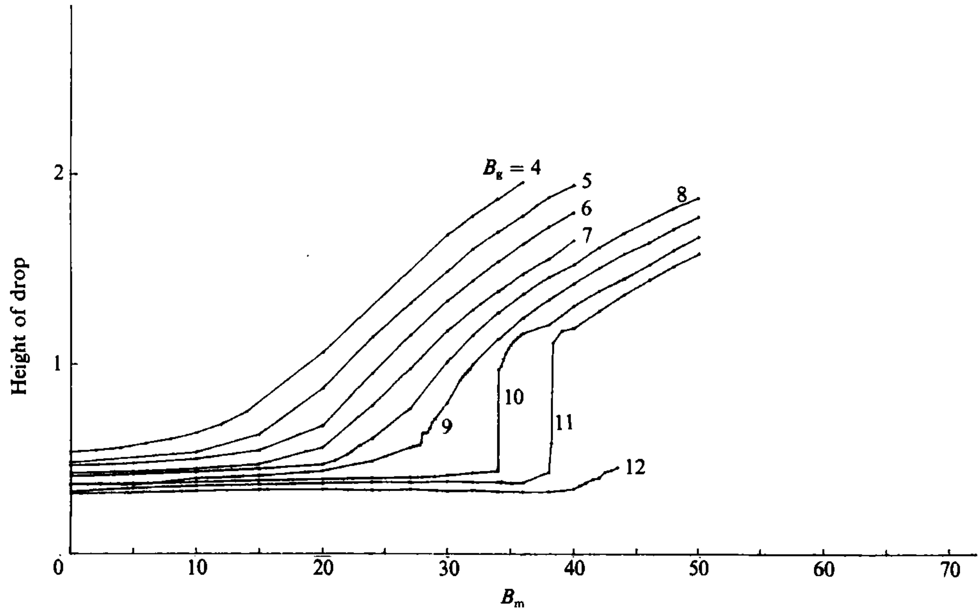

FIGURE 6. Height of a sessile drop versus B_m for different values of B_g.

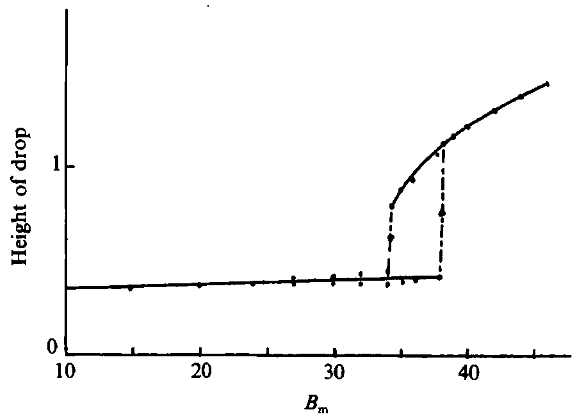

FIGURE 7. Height of a sessile drop at $B_g = 10$ for increasing and decreasing values of B_m.

In figure 7 we plot, at $B_g = 10$, the height of the drop for increasing and decreasing values of B_m. A hysteresis cycle clearly appears.

The number of peaks which appear, see figure 6, and therefore the wavelength depend on the value of B_g. When B_g increases, the drop at $B_m = 0$ becomes flatter and flatter, and looks like a strip, see figure 8. It is tempting to compare the wavelength obtained from a linear analysis for the case of a strip, of thickness e, of ferrofluid lying on a solid plane, see figure 9.

To calculate the interface wavelength for the case of the strip, we shall consider non-dimensional variables. Let u_2^1 and u_2^0 be the upper and lower value of the magnetic potential u_2. The horizontal surface $z = 1$ and the field potentials $u_1 = -\chi_0 z/(1+\chi_0)$, $u_2^1 = -\chi_0/(1+\chi_0)$, $u_2^0 = 0$ are solutions of the free-surface problem. Let us consider now a perturbation $\delta\gamma$ of the interface; the equilibrium equation up to first order is

$$\nabla^2\delta\gamma + B_g\,\delta\gamma - \chi_0 B_m(\delta\boldsymbol{H}\cdot\boldsymbol{k}) = 0, \tag{5.1}$$

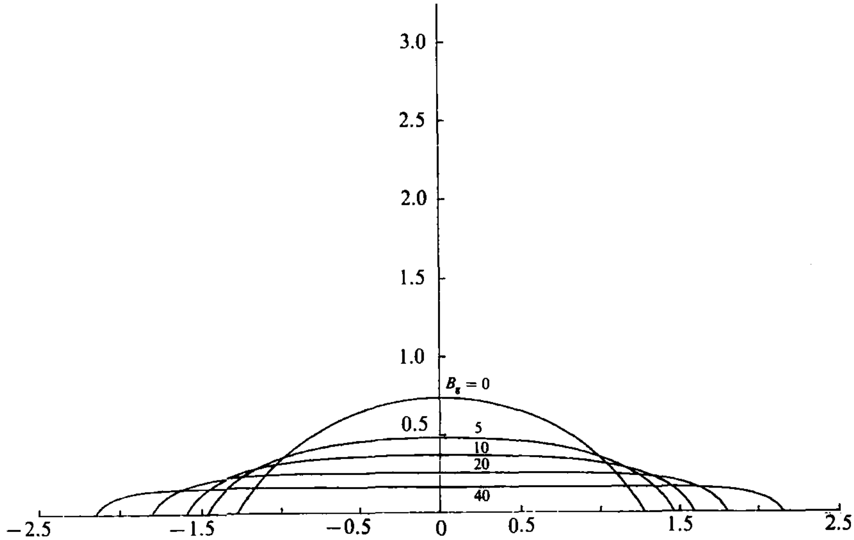

FIGURE 8. Shapes of the drop for B_g increasing and $B_m = 0$.

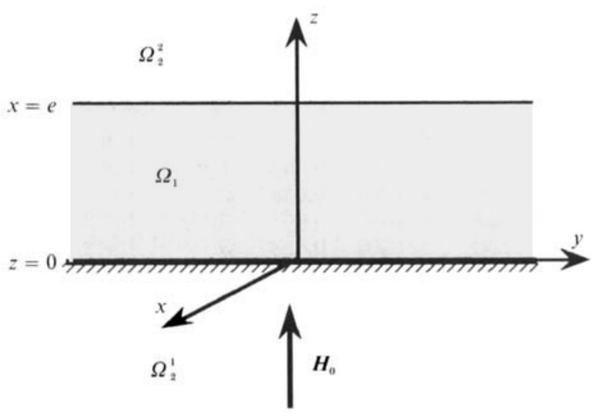

FIGURE 9. A strip of ferrofluid on a solid plane.

with
$$B_g = \frac{\rho g e^2}{\sigma}, \quad B_m = \frac{\mu_0 H_0^2 e}{\sigma}. \tag{5.2}$$

Here $\delta \boldsymbol{H}$ is the perturbation of the magnetic field due to the perturbation of the interface. And it can be shown (Brancher & Séro-Guillaume 1983), that
$$\delta \boldsymbol{H} = \nabla V_1, \tag{5.3}$$
with
$$\nabla^2 V_1 = 0, \quad 0 < z < 1, \tag{5.4}$$
$$\nabla^2 V_2 = 0, \quad z < 0, \quad z > 1, \tag{5.5}$$
$$V_1 - V_2 = -\frac{\chi_0}{1+\chi_0}\delta\gamma, \quad z = 1, \tag{5.6}$$
$$V_1 - V_2 = 0, \quad z = 0, \tag{5.7}$$
$$(1+\chi_0)\frac{\partial V_1}{\partial z} - \frac{\partial V_2}{\partial z} = 0, \quad z = 0, \quad z = 1. \tag{5.8}$$

If $\delta\gamma$ is an eigenfunction of the Laplace operator, i.e. which satisfies the equation
$$\nabla^2(\delta\gamma) = -k^2\,\delta\gamma,$$

B_g	B_m	e	B_g^1	B_m^1	λ_c	λ_l
10	20	0.368	0.829	5.76	0.974	3.178
	24	0.368	0.829	6.91	1.107	3.449
	27	0.368	0.829	7.78	1.194	3.659
	30	0.368	0.829	8.64	1.252	3.870
13	20	0.335	0.892	5.24	1.244	2.649
	30	0.335	0.892	7.86	1.568	3.174
20	20	0.3256	1.295	5.089	2.142	1.99
	25	0.3256	1.295	6.362	2.059	2.157
	35	0.3256	1.295	8.907	2.219	2.504
	40	0.3256	1.295	10.180	2.097	2.684
25	10	0.2259	0.779	1.766	2.196	1.471
	20	0.2259	0.779	3.531	2.333	1.710
	25	0.2259	0.779	4.414	2.394	1.836
	40	0.2259	0.779	7.063	2.436	2.238
30	10	0.1719	0.541	1.343	2.417	1.322
	15	0.1719	0.541	2.015	2.432	1.417
	20	0.1719	0.541	2.687	2.486	1.515
	25	0.1719	0.541	3.359	2.511	1.616
	30	0.1719	0.541	4.031	2.557	1.721
	35	0.1719	0.541	4.702	2.606	1.829
	40	0.1719	0.541	5.374	2.657	1.939
	50	0.1719	0.541	6.718	2.789	2.166
40	10	0.1616	0.638	1.263	1.202	1.125
	15	0.1616	0.638	1.895	1.265	1.195
	20	0.1616	0.638	2.526	1.261	1.268
	25	0.1616	0.638	3.158	1.322	1.343
	30	0.1616	0.638	3.789	1.312	1.420
	35	0.1616	0.638	4.421	1.363	1.501
	40	0.1616	0.638	5.032	1.344	1.528
	48	0.1616	0.638	6.063	1.407	1.716

TABLE 1. Comparison of the wavelengths obtained by the linear analysis and by the direct computation

then the solution of system (5.4)–(5.8) is

$$V_1 = (A_1 \mathrm{e}^{-kz} + B_1 \mathrm{e}^{kz})\,\delta\gamma(x,y), \tag{5.9}$$

$$V_2 = \begin{cases} A_2 \mathrm{e}^{-kz}\,\delta\gamma(x,y), & z > 1 \\ B_2 \mathrm{e}^{-kz}\,\delta\gamma(x,y), & z < 0. \end{cases} \tag{5.10}$$

We can proceed to an analysis in normal modes with $\delta\gamma = \sin \boldsymbol{k}\cdot\boldsymbol{x}$, k being the modulus of $\boldsymbol{k}$ and $\boldsymbol{x}$ the vector position (x,y). A direct calculation shows that the coefficients needed are

$$A_1 = \frac{\chi_0^2 t}{(1+\chi_0)((2+\chi_0)^2 t^2 - \chi_0^2)}, \quad B_1 = \frac{(2+\chi_0)}{\chi_0} A_1, \quad t = \mathrm{e}^k, \tag{5.11}$$

and (5.1) reduces to

$$-k^2 + B_g - B_m \chi_0 (B_1 \mathrm{e}^k - A_1 \mathrm{e}^{-k})\,k = 0. \tag{5.12}$$

The lowest positive solution of (5.12) is the wave vector modulus of the wave which appears at the bifurcation. A comparison is made in table 1, e is the thickness of the drop, with the results given by the computation; B_g^1 and B_m^1 are the new values of the Bond numbers, i.e. when the new length unit is e. We compare λ_c and λ_l, the

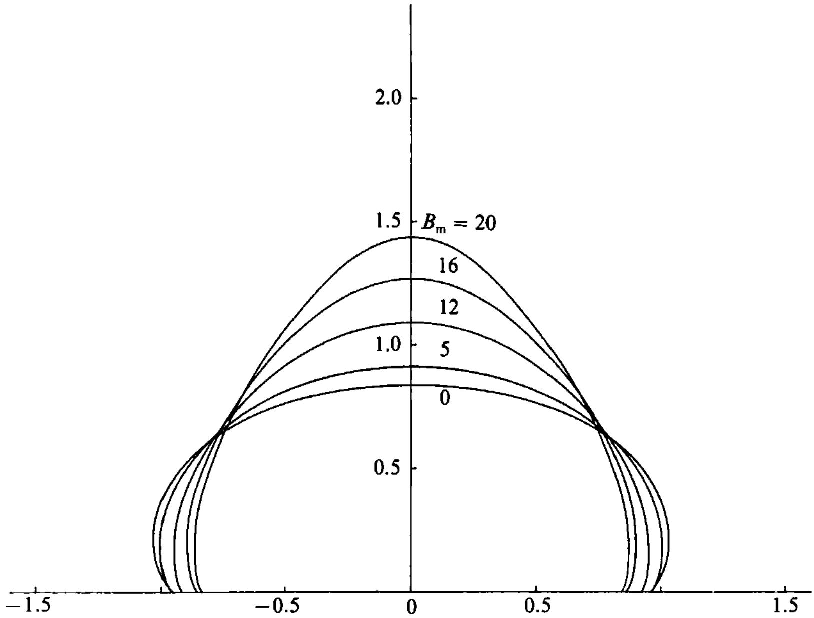

FIGURE 10. Shapes of a non-wetting drop, $\beta = \frac{2}{3}\pi$, for $B_g = 5$ and different values of B_m.

wavelengths obtained respectively by computation and linear analysis. The agreement must become better as B_g increases. The wavelengths agree for $B_g = 20$, 25 and 40 but not for $B_g = 30$. In fact $B_g = 30$ is a transition value for the number of peaks, a third peak appears around this value. Therefore there are boundary effects. As expected the gap between the values of the wavelengths increases when B_m increases.

5.3. *The non-wetting drop*

When $\beta > \frac{1}{2}\pi$, the ferrofluid does not wet the plane. For information we have plotted several equilibrium shapes for different values of B_m, see figure 10.

5.4. *Discussion*

If B_g is about 13 the peaks which have appeared at the interface increase and the centre of the drop falls until it touches the plane; and the algorithm does not converge because, most probably, in that case the drop does not remain axisymmetrical. We suppose that beyond a critical value of B_m the drop will separate in two or several drops.

The modellization of the interactions of the drop with the plane is really simple, and the magnetization law is linear. However, hysteresis appears and is experimentally confirmed, see Brancher & Zouaoui (1987). And thus we can confirm that the method and the algorithm are good and permit successive bifurcations to be followed.

Appendix

Let us consider the problem of the minus sign before E_m in (2.7). We recall that $x = \theta(a, t)$ is the Lagrangian description of the movement of the particles. The aim of this Appendix is to show that the coenergy must be varied with respect to a fixed current. We can consider the magnetization law

$$\boldsymbol{B} = \mu(H)\,\boldsymbol{H} \tag{A 1}$$

or in equivalent form

$$B = \phi(H). \tag{A 2}$$

Relation (A 2) is the one-dimensional form of (A 1), thus in (A 2) B and H can be positive or negative numbers. We suppose that this law is smooth and has an inverse:

$$\boldsymbol{H} = \nu(B)\boldsymbol{B}, \quad H = \psi(B). \qquad \text{(A 3), (A 4)}$$

Let us define magnetic coenergy and energy by

$$E_{\mathrm{m}} = \int_{R^3}\int_0^H \phi(y)\,\mathrm{d}y\,\mathrm{d}\Omega, \quad E_m^* = \int_{R^3}\int_0^B \psi(x)\,\mathrm{d}x\,\mathrm{d}\Omega. \tag{A 5}$$

A variation $\delta\boldsymbol{H}$ in E_{m} produces the variation $\delta E_{\mathrm{m}} = \int_{R^3}\phi(H)\,\delta H\,\mathrm{d}\Omega$ but as B and H are parallel,

$$\mu(H)\,\boldsymbol{H}\cdot\delta\boldsymbol{H} = \tfrac{1}{2}\mu(H)\,\delta\boldsymbol{H}^2 = \mu(H)\,H\delta H = \phi H\,\delta H,$$

taking (A 1) into account we obtain

$$\delta E_{\mathrm{m}} = \int_{R^3} \boldsymbol{B}\cdot\delta\boldsymbol{H}\,\mathrm{d}\Omega$$

which shows that E_{m} is the coenergy. The relation (2.7) is obtained with $\phi(H) = \mu_0(H+M(H))$ in Ω_1 and $\mu_0 H$ in Ω_2. Using the same argument one can show that E_{m}^* is the energy. It is now easy to see, cf. Landau & Lifshitz (1990), that the energy (coenergy) is the Legendre transform of the coenergy (energy), that is

$$E_{\mathrm{m}}^*(B) = \sup_H\left(\left(\int_{R^3}\boldsymbol{B}\cdot\boldsymbol{H}\,\mathrm{d}\Omega - E_{\mathrm{m}}(H)\right)\right).$$

Thus the following relation holds:

$$E_{\mathrm{m}}^*(B) + E_{\mathrm{m}}(H) = \int_{R^3}\boldsymbol{B}\cdot\boldsymbol{H}\,\mathrm{d}\Omega. \tag{A 6}$$

The coenergy and the energy are also functions of θ, the position of the particles,

$$E_{\mathrm{m}}(\boldsymbol{H},\theta), \quad E_{\mathrm{m}}^*(\boldsymbol{B},\theta). \tag{A 7}$$

But in (A 7) the fields $\boldsymbol{B}, \boldsymbol{H}$ and θ are not in fact independent variables, because if the particles are moved the fields lines will be deformed. We must introduce variables linked with the generator. So let us consider the Maxwell equations:

$$\nabla\times\boldsymbol{H} = \boldsymbol{j}_{\mathrm{ex}}, \quad \nabla\cdot\boldsymbol{B} = 0, \qquad \text{(A 8), (A 9)}$$

with appropriate boundary conditions; $\boldsymbol{j}_{\mathrm{ex}}$ is the current density in the inductor. From (A 9) we can consider the vector potential $\boldsymbol{A}$ such that

$$\nabla\times\boldsymbol{A} = \boldsymbol{B}. \tag{A 10}$$

Then from (A 8)–(A 10) it is easy to see that

$$\int_{R^3}\boldsymbol{B}\cdot\boldsymbol{H}\,\mathrm{d}\Omega = \int_{\mathscr{C}}\boldsymbol{A}\cdot\boldsymbol{j}_{\mathrm{ex}}\,\mathrm{d}\Omega, \tag{A 11}$$

where $\mathscr{C}$ is the inductor. The relations $\boldsymbol{j}_{\mathrm{ex}}\to\boldsymbol{H}$ and $\boldsymbol{A}\to\boldsymbol{B}$ can be considered as a change of variables. In (A 6)–(A 7) the energy and coenergy become functions of

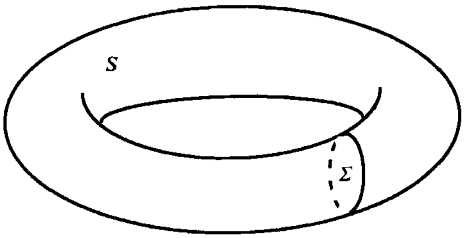

FIGURE 11. Shape of the inductor.

these new variables. This inductor $\mathscr{C}$ is represented, for simplicity, by a torus; the lateral surface of $\mathscr{C}$ is S, and the generator can be modelled by a cutting surface S (the following argument can be easily extended to an inductor with a boundary of any genus, adding as many cutting surfaces as needed), see figure 11.

Following Bossavit (1987) we consider the potential Φ such that:

$$\left.\begin{aligned} \nabla^2 \Phi &= 0 \quad \text{in} \quad \mathscr{C}, \\ \frac{\partial \Phi}{\partial n} &= 0 \quad \text{on} \quad S, \\ [\Phi]_\Sigma &= 1. \end{aligned}\right\} \tag{A 12}$$

$[\Phi]_\Sigma$ is the jump of Φ across Σ. The current density vector $\boldsymbol{j}_{\text{ex}}$ can be uniquely decomposed using the potential Φ:

$$\boldsymbol{j}_{\text{ex}} = I^* \nabla \Phi + \boldsymbol{j}'; \tag{A 13}$$

$\boldsymbol{j}'$ verifies the relations: $\boldsymbol{j}' = \nabla \times \boldsymbol{\alpha}$ in $\mathscr{C}$ and $\boldsymbol{\alpha} \times \boldsymbol{n} = 0$ on S, and it is orthogonal to $\nabla \Phi$, that is

$$\int_{\mathscr{C}} \nabla \Phi \cdot \boldsymbol{j}' \, \mathrm{d}\Omega = 0. \tag{A 14}$$

The same decomposition holds for the vector potential $\boldsymbol{A}$:

$$\boldsymbol{A} = U^* \nabla \Phi + \boldsymbol{A}'. \tag{A 15}$$

Taking (A 14) into account for $\boldsymbol{j}'$ and $\boldsymbol{A}'$, we can write

$$\int_{\mathscr{C}} \boldsymbol{A} \cdot \boldsymbol{j}_{\text{ex}} \, \mathrm{d}\Omega = I^* U^* \int_{\mathscr{C}} (\nabla \Phi)^2 \, \mathrm{d}\Omega + \int_{\mathscr{C}} \boldsymbol{A}' \cdot \boldsymbol{j}' \, \mathrm{d}\Omega. \tag{A 16}$$

Note that the integral $\int_{\Sigma'} (\partial \Phi / \partial n) \, \mathrm{d}S$ is constant for any surface Σ' cutting $\mathscr{C}$ (but different from the surface Σ). But I, the total current in the inductor, is given by

$$I = \int_{\Sigma'} \boldsymbol{j}_{\text{ex}} \cdot \boldsymbol{n} \, \mathrm{d}S = I^* \int_{\Sigma'} \frac{\partial \Phi}{\partial n} \, \mathrm{d}S. \tag{A 17}$$

Therefore, if I is given, I^* is known by (A 17), and the field $\boldsymbol{j}'$ in (A 13) plays no role and can be set to 0. And then (A 16) reduces to

$$\int_{\mathscr{C}} \boldsymbol{A} \cdot \boldsymbol{j}_{\text{ex}} \, \mathrm{d}\Omega = I^* U^* \int_{\mathscr{C}} (\nabla \Phi)^2 \, \mathrm{d}\Omega. \tag{A 18}$$

Once I^*, U^* are given, $\boldsymbol{j}_{\text{ex}}$ and $\boldsymbol{A}$ are known and then $\boldsymbol{H}$ and $\boldsymbol{B}$ can be uniquely determined; U^* is a flux such that its time derivative can be considered as the tension of the generator. For simplicity we can call $I^* \int_{\mathscr{C}} (\nabla \Phi)^2 \,\mathrm{d}\Omega$ again I^*. With these new independent variables (A 6) can be written

$$E_{\mathrm{m}}^*(U^*, \theta) + E_{\mathrm{m}}(I^*, \theta) = U^* I^*. \tag{A 19}$$

The Legendre transformation can be performed with U^* and I^*. Then

$$\mathrm{d}E_{\mathrm{m}}(I^*, \theta) = \frac{\partial E_{\mathrm{m}}}{\partial I^*} \mathrm{d}I^* + \frac{\partial E_{\mathrm{m}}}{\partial \theta} \delta\theta = k\, \mathrm{d}I^* + \boldsymbol{F} \cdot \delta\boldsymbol{\theta}. \tag{A 20}$$

From (A 19), (A 20) (and taking into account the remark at the end of §2) it is clear that to obtain the force we must vary the coenergy with respect to the position with a fixed external current, or vary minus the energy with a fixed flux.

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